

# The limit distribution of ratios of jumps and sums of jumps of subordinators

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## Abstract

Let  $V_t$  be a driftless subordinator, and let denote  $m_t^{(1)} \geq m_t^{(2)} \geq \dots$  its jump sequence on interval  $[0, t]$ . Put  $V_t^{(k)} = V_t - m_t^{(1)} - \dots - m_t^{(k)}$  for the  $k$ -trimmed subordinator. In this note we characterize under what conditions the limiting distribution of the ratios  $V_t^{(k)}/m_t^{(k+1)}$  and  $m_t^{(k+1)}/m_t^{(k)}$  exist, as  $t \downarrow 0$  or  $t \rightarrow \infty$ .

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## 1 Introduction and results

Let  $V_t$ ,  $t \geq 0$ , be a subordinator with Lévy measure  $\Lambda$  and drift 0. Its Laplace transform is given by

$$\mathbf{E}e^{-\lambda V_t} = \exp \left\{ -t \int_0^\infty (1 - e^{-\lambda v}) \Lambda(dv) \right\},$$

where the Lévy measure  $\Lambda$  satisfies

$$\int_0^\infty \min\{1, x\} \Lambda(dx) < \infty. \quad (1)$$

Put  $\bar{\Lambda}(x) = \Lambda((x, \infty))$ . Then  $\bar{\Lambda}(x)$  is nonincreasing and right continuous on  $(0, \infty)$ . When  $t \downarrow 0$  we also assume that  $\bar{\Lambda}(0+) = \infty$ , which is necessary and sufficient to assure that there is an infinite number of jumps up to time  $t$ , for any  $t > 0$ .

Denote  $m_t^{(1)} \geq m_t^{(2)} \geq \dots$  the ordered jumps of  $V_s$  up to time  $t$ , and for  $k \geq 0$  consider the trimmed subordinator

$$V_t^{(k)} = V_t - \sum_{j=1}^k m_t^{(j)}.$$

We investigate the asymptotic distribution of jump sizes as  $t \downarrow 0$  and  $t \rightarrow \infty$ . Specifically, we shall determine a necessary and sufficient condition in terms of the Lévy measure  $\Lambda$  for the convergence in distribution of the ratios  $V_t^{(k)}/m_t^{(k+1)}$  and  $m_t^{(k+1)}/m_t^{(k)}$ . Observe in this notation that  $V_t^{(0)} = V_t$  is the subordinator and  $m_t^{(1)}$  is the largest jump.

An extended random variable  $W$  can take the value  $\infty$  with positive probability, in which case  $W$  has a defective distribution function  $F$ , meaning that  $F(\infty) < 1$ . We shall call an extended random variable proper, if it is finite a.s. In this case its  $F$  is a probability distribution, i.e.  $F(\infty) = 1$ . Here we are using the language of the definition given on p. 127 of Feller [8].

**Theorem 1.** *For any choice of  $k \geq 0$  the ratio  $V_t^{(k)}/m_t^{(k+1)}$  converges in distribution to an extended random variable  $W_k$  as  $t \downarrow 0$  ( $t \rightarrow \infty$ ) if and only if one of the following holds:*

- (i)  $\bar{\Lambda}$  is regularly varying at 0 ( $\infty$ ) with parameter  $-\alpha$ ,  $\alpha \in (0, 1)$ , in which case  $W_k$  is a proper random variable with Laplace transform

$$g_k(\lambda) = \frac{e^{-\lambda}}{\left[1 + \alpha \int_0^1 (1 - e^{-\lambda y}) y^{-\alpha-1} dy\right]^{k+1}}; \quad (2)$$

- (ii)  $\bar{\Lambda}$  is slowly varying at 0 ( $\infty$ ), in which case  $W_k = 1$  a.s.;

- (iii) the condition

$$\frac{x\bar{\Lambda}(x)}{\int_0^x u\Lambda(du)} \rightarrow 0 \quad \text{as } x \downarrow 0 \text{ (} x \rightarrow \infty \text{)} \quad (3)$$

holds, in which case  $V_t^{(k)}/m_t^{(k+1)} \xrightarrow{\mathbf{P}} \infty$ , that is  $W_k = \infty$  a.s.

Note that Theorem 1 says that the situation  $0 < \mathbf{P}\{W_k = \infty\} < 1$  cannot happen.

The corresponding problem for nonnegative i.i.d. random variables was investigated by Darling [6] and Breiman [4], in the  $k = 0$  case. In this case Darling proved the sufficiency parts corresponding to (i) and (ii) (Theorem 5.1 and Theorem 3.2 in [6]), in particular the limit  $W_0$  has the same distribution as given by Darling in his Theorem 5.1, while Breiman proved the necessity parts corresponding to (i), (ii) and (iii) (Theorem 3 (p. 357), Theorem 2 and Theorem 4 in [4]). A special case of Theorem 1 in Teugels [12] gives the sufficiency analog of (i) in the case of i.i.d. nonnegative sums for any  $k \geq 0$ .

The necessary and sufficient condition in the cases (ii) and (iii), stated in the more general setup of Lévy processes without a normal component, is given by Buchmann, Fan and Maller [5].

Next we shall investigate the asymptotic distribution of the ratio of two consecutive ordered jumps  $m_t^{(k+1)}/m_t^{(k)}$ ,  $k \geq 1$ . We shall obtain the analog for subordinators of a special case of a result that Bingham and Teugels [3] established for i.i.d. nonnegative random variables. This will follow from a general result on the asymptotic distribution of ratios of the form defined for  $k \geq 1$  by

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)}, t > 0,$$

where for each  $k \geq 1$ ,  $S_k = \omega_1 + \dots + \omega_k$ , with  $\omega_1, \omega_2, \dots$  being i.i.d. mean 1 exponential random variables and  $\psi$  is the nonincreasing and right continuous function defined for  $s > 0$  by

$$\psi(s) = \sup\{y : \bar{\Pi}(y) > s\},$$

with  $\Pi$  being a positive measure on  $(0, \infty)$  such that  $\bar{\Pi}(x) = \Pi((x, \infty)) \rightarrow 0$ , as  $x \rightarrow \infty$ . Note that we do not require  $\Pi$  to be a Lévy measure. Also whenever we consider the asymptotic distribution of  $r_k(t)$  as  $t \downarrow 0$  we shall assume that  $\bar{\Pi}(0+) = \infty$ .

We call a function  $f$  *rapidly varying at 0* with index  $-\infty$ ,  $f \in \text{RV}_0(-\infty)$ , if

$$\lim_{x \downarrow 0} \frac{f(\lambda x)}{f(x)} = \begin{cases} 0, & \text{for } \lambda > 1, \\ 1, & \text{for } \lambda = 1, \\ \infty, & \text{for } \lambda < 1. \end{cases}$$

Correspondingly, a function  $f$  is *rapidly varying at  $\infty$*  with index  $-\infty$ ,  $f \in \text{RV}_\infty(-\infty)$ , if the same holds with  $x \rightarrow \infty$ .

**Theorem 2.** *For any choice of  $k \geq 1$  the ratio  $r_k(t)$  converges in distribution as  $t \downarrow 0$  ( $t \rightarrow \infty$ ) to a random variable  $Y_k$  if and only if one of the following holds:*

- (i)  $\bar{\Pi}$  is regularly varying at 0 ( $\infty$ ) with parameter  $-\alpha \in (-\infty, 0)$ , in which case  $Y_k$  has the  $\text{Beta}(k\alpha, 1)$  distribution, i.e.

$$G_k(x) = \mathbf{P}\{Y_k \leq x\} = x^{k\alpha}, \quad x \in [0, 1]; \quad (4)$$

- (ii)  $\bar{\Pi}$  is slowly varying at 0 ( $\infty$ ), in which case  $Y_k = 0$  a.s.

- (iii)  $\bar{\Pi}$  is rapidly varying at 0 ( $\infty$ ) with index  $-\infty$ , in which case  $Y_k = 1$  a.s.

Theorem 2 has some important applications to the asymptotic distribution of the ratio of two consecutive ordered jumps  $m_t^{(k+1)}/m_t^{(k)}$ ,  $k \geq 1$ , of a Lévy process. Let  $X_t$ ,  $t \geq 0$ , be a Lévy process whose Lévy measure  $\Lambda$  is concentrated on  $(0, \infty)$ . Here in addition to  $\bar{\Lambda}(x) \rightarrow 0$  as  $x \rightarrow \infty$ , we require that

$$\int_0^\infty \min\{1, x^2\} \Lambda(dx) < \infty. \quad (5)$$

In this setup one has the distributional representation for  $k \geq 1$

$$\left(m_t^{(k)}, m_t^{(k+1)}\right) \stackrel{\mathcal{D}}{=} (\varphi(S_k/t), \varphi(S_{k+1}/t)), \quad (6)$$

with  $\varphi$  defined for  $s > 0$  to be

$$\varphi(s) = \sup\{y : \bar{\Lambda}(y) > s\}. \quad (7)$$

It is readily checked that  $\varphi$  is nonincreasing and right continuous. Moreover, whenever  $\Lambda$  is the Lévy measure of a subordinator  $V_t$ , condition (1) holds, which is equivalent to

$$\int_\delta^\infty \varphi(s) ds < \infty, \text{ for any } \delta > 0. \quad (8)$$

The distributional representation in (6) follows from Proposition 1 in Kevei and Mason [7]. See the proof of Theorem 2 below, while for general spectrally positive Lévy processes it can be deduced using the same methods that Maller and Mason [9] derived the distributional representation for a Lévy process given in their Proposition 5.7.

When applying Theorem 2 to the asymptotic distribution of consecutive ordered jumps at 0 or  $\infty$  of a Lévy processes  $X_t$  whose Lévy measure  $\Lambda$  is concentrated on  $(0, \infty)$ , we have to keep in mind that (5) must always hold and (1) must be satisfied whenever  $X_t$  is a subordinator. For instance in the case of a subordinator  $V_t$ , whenever  $m_t^{(k+1)}/m_t^{(k)}$  converges in distribution to a random variable  $Y_k$  as  $t \downarrow 0$ , Theorem 2 says that  $\bar{\Lambda}$  is regularly varying at 0. Further since (1) must hold, the parameter  $-\alpha$  is necessarily be in  $[-1, 0]$ , while there is no such restriction when considering convergence in distribution as  $t \rightarrow \infty$ .

In the special case when  $V_t$  is an  $\alpha$ -stable subordinator,  $\alpha \in (0, 1)$ , and  $m^{(1)} > m^{(2)} > \dots$  is its jump sequence on  $[0, 1]$ , then  $(m^{(1)}/V_1, m^{(2)}/V_1, \dots)$  has the Poisson–Dirichlet law with parameter  $(\alpha, 0)$  ( $\text{PD}(\alpha, 0)$ ). See Bertoin [1] p. 90. The ratio of the  $(k+1)^{\text{th}}$  and  $k^{\text{th}}$  element of a vector, which has the  $\text{PD}(\alpha, 0)$  law, has the Beta( $k\alpha, 1$ ) distribution (Proposition 2.6 in [1]).

## 2 Proofs

In the proofs we only consider the case when  $t \downarrow 0$ , as the  $t \rightarrow \infty$  case is nearly identical.

### 2.1 Proof of Theorem 1

First we calculate the Laplace exponent of the ratio using the notation  $\varphi$  defined in (7). We see by the nonincreasing version of the change of variables formula stated in (4.9) Proposition of Revuz and Yor [10], which is given in Lemma 1 in [7],

$$\begin{aligned} \mathbf{E}e^{-\lambda V_t} &= \exp \left\{ -t \int_0^\infty (1 - e^{-\lambda v}) \Lambda(\mathrm{d}v) \right\} \\ &= \exp \left\{ -t \int_0^\infty (1 - e^{-\lambda \varphi(x)}) \mathrm{d}x \right\}. \end{aligned}$$

The key ingredient of our proofs is a distributional representation of the subordinator  $V_t$  given in Kevei and Mason (Proposition 1 in [7]), which follows from a general representation by Rosiński [11]. It states that for  $t > 0$

$$V_t \stackrel{\mathcal{D}}{=} \sum_{i=1}^{\infty} \varphi \left( \frac{S_i}{t} \right). \quad (9)$$

From the proof of this result it is clear that  $\varphi(S_i/t)$  corresponds to  $m_t^{(i)}$ , for  $i \geq 1$ . Therefore

$$\frac{V_t^{(k)}}{m_t^{(k+1)}} \stackrel{\mathcal{D}}{=} \frac{\sum_{i=k+1}^{\infty} \varphi(S_i/t)}{\varphi(S_{k+1}/t)}.$$

Conditioning on  $S_{k+1} = s$  and using the independence we can write

$$\begin{aligned} \sum_{i=k+2}^{\infty} \varphi(S_i/t) &= \sum_{i=k+2}^{\infty} \varphi\left(\frac{s}{t} + \frac{S_i - s}{t}\right) \\ &\stackrel{\mathcal{D}}{=} \sum_{i=1}^{\infty} \varphi\left(\frac{s}{t} + \frac{S_i}{t}\right) \\ &= \sum_{i=1}^{\infty} \varphi_{s/t}(S_i/t), \end{aligned}$$

where  $\varphi_s(x) = \varphi(s+x)$ . Note that the latter sum has the same form as in (9), therefore it is equal in distribution to a subordinator  $V^{(s/t)}(t)$  with Laplace transform

$$\begin{aligned} \mathbf{E}e^{-\lambda V_t^{(s/t)}} &= \exp\left\{-t \int_0^{\infty} (1 - e^{-\lambda \varphi_{s/t}(x)}) dx\right\} \\ &= \exp\left\{-t \int_{s/t}^{\infty} (1 - e^{-\lambda \varphi(x)}) dx\right\}. \end{aligned} \tag{10}$$

Now we can compute the Laplace transform of the ratio  $V_t^{(k)}/m_t^{(k+1)}$ . Since  $S_{k+1}$  has Gamma( $k+1, 1$ ) distribution, the law of total probability and (10) give

$$\begin{aligned} \mathbf{E}e^{-\lambda \frac{V_t^{(k)}}{m_t^{(k+1)}}} &= \mathbf{E}e^{-\lambda \frac{\sum_{i=k+1}^{\infty} \varphi(S_i/t)}{\varphi(S_{k+1}/t)}} \\ &= \int_0^{\infty} \frac{s^k}{k!} e^{-s} \left[ e^{-\lambda} \mathbf{E}e^{-\frac{\lambda}{\varphi(s/t)} \sum_{i=1}^{\infty} \varphi_{s/t}(S_i/t)} \right] ds \\ &= e^{-\lambda} \int_0^{\infty} \frac{s^k}{k!} e^{-s} \exp\left\{-t \int_{s/t}^{\infty} [1 - e^{-\lambda \frac{\varphi(x)}{\varphi(s/t)}}] dx\right\} ds \\ &= \frac{t^{k+1}}{k!} e^{-\lambda} \int_0^{\infty} u^k \exp\left\{-t \left(u + \int_u^{\infty} [1 - e^{-\lambda \frac{\varphi(x)}{\varphi(u)}}] dx\right)\right\} du \\ &= \frac{t^{k+1}}{k!} e^{-\lambda} \int_0^{\infty} u^k e^{-t\Psi(u, \lambda)} du, \end{aligned} \tag{11}$$

where

$$\Psi(u, \lambda) = u + \int_u^\infty [1 - e^{-\lambda \frac{\varphi(x)}{\varphi(u)}}] dx. \quad (12)$$

Since  $\varphi$  is right continuous on  $(0, \infty)$ ,  $\Psi(\cdot, \lambda)$  is also right continuous on  $(0, \infty)$ . Further a short calculation shows that this function is strictly increasing for any  $\lambda > 0$ , moreover for  $u_1 > u_2$

$$\Psi(u_1, \lambda) - \Psi(u_2, \lambda) \geq e^{-\lambda}(u_1 - u_2).$$

Clearly  $\Psi(0, \lambda) = 0$  and  $\Psi(\infty, \lambda) = \infty$ . Therefore

$$\Psi_k(\cdot, \lambda) := \Psi\left(\left((k+1)\cdot\right)^{1/(k+1)}, \lambda\right)$$

has a right continuous increasing inverse function given by

$$Q_\lambda(s) = \inf\{v : \Psi_k(v, \lambda) > s\}, \text{ for } s \geq 0,$$

such that  $Q_\lambda(0) = 0$  and  $\lim_{x \rightarrow \infty} Q_\lambda(x) = \infty$ . (For the right continuity part see (4.8) Lemma in Revuz and Yor [10].)

**Necessity.** Assuming that  $V_t^{(k)}/m_t^{(k+1)}$  converges in distribution as  $t \rightarrow 0$  to some extended random variable  $W_k$ , we can apply Theorem 2a on p. 210 of Feller [8] to conclude that its Laplace transform also converges, i.e.

$$\begin{aligned} \int_0^\infty u^k e^{-t\Psi(u, \lambda)} du &= \int_0^\infty e^{-t\Psi_k(v, \lambda)} dv \\ &= \int_0^\infty e^{-ty} dQ_\lambda(y) \sim \frac{e^\lambda g_k(\lambda) k!}{t^{k+1}}, \text{ as } t \rightarrow 0, \end{aligned}$$

where  $g_k(\lambda) = \mathbf{E}e^{-\lambda W_k}$ , and  $W_k$  can possibly have a defective distribution, i.e. possibly  $\mathbf{P}\{W_k = \infty\} > 0$ . (Here we used the change of variables formula given in (4.9) Proposition in Revuz and Yor [10].) By Karamata's Tauberian theorem (Theorem 1.7.1 in [2])

$$Q_\lambda(y) \sim \frac{y^{k+1}}{k+1} e^\lambda g_k(\lambda), \quad \text{as } y \rightarrow \infty,$$

and thus by Theorem 1.5.12 in [2]

$$\Psi_k(v, \lambda) \sim \left(\frac{(k+1)v}{e^\lambda g_k(\lambda)}\right)^{1/(k+1)}, \quad \text{as } v \rightarrow \infty,$$

and hence

$$\Psi(u, \lambda) \sim u \left[ e^\lambda g_k(\lambda) \right]^{-\frac{1}{k+1}}, \quad \text{as } u \rightarrow \infty.$$

Substituting back into (12) we obtain for any  $\lambda > 0$

$$\lim_{u \rightarrow \infty} \frac{1}{u} \int_u^\infty \left( 1 - e^{-\lambda \frac{\varphi(x)}{\varphi(u)}} \right) dx = \left[ e^\lambda g_k(\lambda) \right]^{-\frac{1}{k+1}} - 1. \quad (13)$$

Note that the limit  $W_k$  is  $\geq 1$ , with probability 1, and so  $g_k(\lambda) \leq e^{-\lambda}$ . Thus for any  $\lambda$

$$\left[ e^\lambda g_k(\lambda) \right]^{-\frac{1}{k+1}} - 1 \geq 0.$$

For any  $x \geq 0$  we have  $1 - e^{-x} \leq x$ . Therefore by (13) we obtain for any  $\lambda > 0$

$$\liminf_{u \rightarrow \infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x) dx \geq \frac{1}{\lambda} \left( \left[ e^\lambda g_k(\lambda) \right]^{-\frac{1}{k+1}} - 1 \right). \quad (14)$$

On the other hand, by monotonicity  $\varphi(x)/\varphi(u) \leq 1$  for  $u \leq x$ . Therefore for any  $1 > \varepsilon > 0$  there exists a  $\lambda_\varepsilon > 0$ , such that for all  $0 < \lambda < \lambda_\varepsilon$

$$1 - e^{-\lambda \frac{\varphi(x)}{\varphi(u)}} \geq (1 - \varepsilon) \frac{\lambda \varphi(x)}{\varphi(u)}, \quad \text{for } x \geq u.$$

Using again (13) and keeping (8) in mind, this implies that for such  $\lambda$

$$\limsup_{u \rightarrow \infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x) dx \leq \frac{1}{1 - \varepsilon} \frac{1}{\lambda} \left( \left[ e^\lambda g_k(\lambda) \right]^{-\frac{1}{k+1}} - 1 \right). \quad (15)$$

In particular, we obtain that, whenever  $g_k(\lambda) \neq 0$  (i.e.  $\mathbf{P}\{W_k < \infty\} > 0$ )

$$0 \leq \liminf_{u \rightarrow \infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x) dx \leq \limsup_{u \rightarrow \infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x) dx < \infty.$$

Note that in (14) the greatest lower bound is 0 for all  $\lambda > 0$  if and only if  $g_k(\lambda) = e^{-\lambda}$ , in which case  $W_k = 1$ . Then the upper bound for the limsup in (15) is 0, thus

$$\lim_{u \rightarrow \infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x) dx = 0,$$

which by Proposition 2.6.10 in [2] applied to the function  $f(x) = x\varphi(x)$  implies that  $\varphi \in \text{RV}_\infty(-\infty)$ , and so, by Theorem 2.4.7 in [2],  $\bar{\Lambda}$  is slowly varying at 0. We have proved that  $W_k = 1$  if and only if  $\bar{\Lambda}$  is slowly varying at 0.



In the following we assume that  $\mathbf{P}\{W_k > 1\} > 0$ , therefore the liminf in (14) is strictly positive. Let

$$a = \liminf_{\lambda \downarrow 0} \frac{1}{\lambda} \left( \left[ e^\lambda g_k(\lambda) \right]^{-\frac{1}{k+1}} - 1 \right) \leq \limsup_{\lambda \downarrow 0} \frac{1}{\lambda} \left( \left[ e^\lambda g_k(\lambda) \right]^{-\frac{1}{k+1}} - 1 \right) = b.$$

By (15) and (14),  $a > 0$  and  $b < \infty$ . Moreover

$$b \leq \liminf_{u \rightarrow \infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x) dx \leq \limsup_{u \rightarrow \infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x) dx \leq a,$$

which forces

$$a = b = \lim_{u \rightarrow \infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x) dx = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left( \left[ e^\lambda g_k(\lambda) \right]^{-\frac{1}{k+1}} - 1 \right).$$

By Karamata's theorem (Theorem 1.6.1 (ii) in [2]) we obtain that  $\varphi$  is regularly varying at infinity with parameter  $-a^{-1} - 1 =: -\alpha^{-1}$ , so  $\Lambda$  is regularly varying with parameter  $-\alpha$  at zero with  $\alpha \in (0, 1)$ .

Let us consider the case when  $W_k = \infty$  a.s., that is  $V_t^{(k)}/m_t^{(k+1)} \xrightarrow{\mathbf{P}} \infty$ . All the previous computations are valid, with  $g_k(\lambda) = \mathbf{E}e^{-\lambda\infty} \equiv 0$ . Thus, from (14) we have

$$\lim_{u \rightarrow \infty} \frac{1}{u\varphi(u)} \int_u^\infty \varphi(x) dx = \infty.$$

From this, through the change of variables formula we obtain (3).

**Sufficiency and the limit.** Consider first the special case when  $\varphi(x) = x^{-\frac{1}{\alpha}}$ ,  $\alpha \in (0, 1)$ . Then a quick calculation gives

$$\frac{1}{u} \int_u^\infty \left( 1 - e^{-\lambda \frac{\varphi(x)}{\varphi(u)}} \right) dx = \alpha \int_0^1 \left( 1 - e^{-\lambda y} \right) y^{-\alpha-1} dy.$$

By formula (13) for the Laplace transform of the limit we obtain (2).

The sufficiency can be proved by standard arguments for regularly varying functions. Using Potter bounds (Theorem 1.5.6 in [2]) one can show that for  $\alpha \in (0, 1)$

$$\lim_{u \rightarrow \infty} \frac{1}{u} \Psi(u, \lambda) = 1 + \alpha \int_0^1 \left( 1 - e^{-\lambda y} \right) y^{-\alpha-1} dy,$$

from which, through formula (11), the convergence readily follows. As already mentioned, cases (ii) and (iii) are treated in [5].

## 2.2 Proof of Theorem 2

Using that  $\psi(s) \leq x$  if and only if  $\bar{\Pi}(x) \leq s$ , for the distribution function of the ratio we have for  $x \in (0, 1)$

$$\begin{aligned}
\mathbf{P}\{r_k(t) \leq x\} &= \mathbf{P}\left\{\frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \leq x\right\} \\
&= \int_0^\infty \frac{s^{k-1}}{(k-1)!} e^{-s} \mathbf{P}\left\{\psi\left(\frac{s+S_1}{t}\right) \leq x\psi\left(\frac{s}{t}\right)\right\} ds \\
&= \int_0^\infty \frac{s^{k-1}}{(k-1)!} e^{-s} e^{-[t\bar{\Pi}(x\psi(s/t))-s]} ds \\
&= \frac{t^k}{(k-1)!} \int_0^\infty u^{k-1} e^{-t\bar{\Pi}(x\psi(u))} du.
\end{aligned} \tag{16}$$

**Necessity.** Assume that the limit distribution function  $G_k$  exists. Write

$$\frac{t^k}{(k-1)!} \int_0^\infty u^{k-1} e^{-t\bar{\Pi}(x\psi(u))} du = \frac{t^k}{(k-1)!} \int_0^\infty e^{-t\Phi(v,x)} dv,$$

where  $\Phi(\cdot, x) = \bar{\Pi}\left(x\psi((k\cdot)^{1/k})\right)$ . Note that for each  $x \in (0, 1)$  the function  $\Phi(\cdot, x)$  is monotone nonincreasing and right continuous, since  $\bar{\Pi}$  and  $\psi$  are both monotone nonincreasing and right continuous. Let

$$\mathcal{G}_k = \{x : x \text{ is a continuity point of } G_k \text{ in } (0, 1) \text{ such that } G_k(x) > 0\}.$$

First assume that  $\mathbf{P}\{Y_k < 1\} > 0$ . Clearly we can now proceed as in the proof of Theorem 1 to apply Karamata's Tauberian theorem (Theorem 1.7.1 in [2]) to give that for any  $x \in \mathcal{G}_k$ ,

$$\lim_{u \rightarrow \infty} \frac{\bar{\Pi}(x\psi(u))}{u} = [G_k(x)]^{-\frac{1}{k}}. \tag{17}$$

We claim that (17) implies the regular variation of  $\bar{\Pi}$ . When  $\bar{\Pi}$  is continuous and strictly decreasing we get by changing variables to  $\psi(u) = t$ ,  $u = \bar{\Pi}(t)$ , that we have for any  $x \in \mathcal{G}_k$

$$\lim_{t \downarrow 0} \frac{\bar{\Pi}(tx)}{\bar{\Pi}(t)} = [G_k(x)]^{-\frac{1}{k}},$$

which by an easy application of Proposition 1.10.5 in [2] implies that  $\bar{\Pi}$  is regularly varying.

Note that the jumps of  $\overline{\Pi}$  correspond to constant parts of  $\psi$ , and vice versa. Put  $\mathcal{J} = \{z : \overline{\Pi}(z-) > \overline{\Pi}(z)\}$  for the jump points of  $\overline{\Pi}$ . For  $z \in \mathcal{J}$  and  $y \in [\overline{\Pi}(z), \overline{\Pi}(z-))$  we have  $\psi(y) = z$ . Substituting into (17) we have

$$\lim_{z \downarrow 0, z \in \mathcal{J}} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = [G_k(x)]^{-\frac{1}{k}}, \quad \text{and} \quad \lim_{z \downarrow 0, z \in \mathcal{J}} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z-)} = [G_k(x)]^{-\frac{1}{k}}. \quad (18)$$

To see how the second limit holds in (18) note that for any  $0 < \varepsilon < 1$  and  $z \in \mathcal{J}$ , we have  $\psi(\varepsilon \overline{\Pi}(z) + (1 - \varepsilon) \overline{\Pi}(z-)) = z$  and thus

$$\lim_{z \downarrow 0, z \in \mathcal{J}} \frac{\overline{\Pi}(xz)}{\varepsilon \overline{\Pi}(z) + (1 - \varepsilon) \overline{\Pi}(z-)} = [G_k(x)]^{-\frac{1}{k}}.$$

Since  $0 < \varepsilon < 1$  can be chosen arbitrarily close to 0 this implies the validity of the second limit in (18). Therefore by choosing any  $x \in \mathcal{G}_k$  we get

$$\lim_{z \downarrow 0} \frac{\overline{\Pi}(z-)}{\overline{\Pi}(z)} = 1. \quad (19)$$

Let

$$\mathcal{A} = \{z > 0 : \overline{\Pi}(z - \varepsilon) > \overline{\Pi}(z) \text{ for all } z > \varepsilon > 0\}.$$

This set contains exactly those points  $z$  for which  $\psi(\overline{\Pi}(z)) = z$ . With this notation formula (17) can be written as

$$\lim_{z \downarrow 0, z \in \mathcal{A}} \frac{\overline{\Pi}(xz)}{\overline{\Pi}(z)} = [G_k(x)]^{-\frac{1}{k}}, \text{ for } x \in \mathcal{G}_k. \quad (20)$$

This together with (19) will allow us to apply Proposition 1.10.5 in [2] to conclude that  $\overline{\Pi}$  is regularly varying. We shall need the following technical lemma.

**Lemma 1.** *Whenever (19) holds, there exists a strictly decreasing sequence  $z_n \in \mathcal{A}$  such that  $z_n \rightarrow 0$  and*

$$\lim_{n \rightarrow \infty} \frac{\overline{\Pi}(z_{n+1})}{\overline{\Pi}(z_n)} = 1. \quad (21)$$

*Proof.* Choose  $z_1 \in \mathcal{A}$  such that  $\overline{\Pi}(z_1) > 0$ , and define for each  $n \geq 1$

$$z_{n+1} = \sup \left\{ z > 0 : \overline{\Pi}(z) > \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-) \right\}.$$

Notice that the sequence  $\{z_n\}$  is well-defined, since  $\overline{\Pi}(0+) = \infty$  and it is decreasing. Further we have

$$\overline{\Pi}(z_{n+1}-) \geq \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-) \text{ and } \overline{\Pi}(z_{n+1}) \leq \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-),$$

where the second inequality follows by right continuity of  $\overline{\Pi}$ . Also note that  $z_{n+1} < z_n$ , since otherwise if  $z_{n+1} = z_n$ , then

$$\overline{\Pi}(z_{n+1}-) = \overline{\Pi}(z_n-) > \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-),$$

which is impossible. Observe that each  $z_{n+1}$  is in  $\mathcal{A}$  since by the definition of  $z_{n+1}$  for all  $0 < \varepsilon < z_{n+1}$

$$\overline{\Pi}(z_{n+1} - \varepsilon) > \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-) \geq \overline{\Pi}(z_{n+1}).$$

Clearly since  $\{z_n\}$  is a decreasing and positive sequence,  $\lim_{n \rightarrow \infty} z_n = z^*$  exists and is  $\geq 0$ . By construction

$$\overline{\Pi}(z_{n+1}-) \geq \left(1 + \frac{1}{n}\right) \overline{\Pi}(z_n-) \geq \prod_{k=1}^n \left(1 + \frac{1}{k}\right) \overline{\Pi}(z_1-).$$

The infinite product  $\prod_{n=1}^{\infty} (1 + 1/n) = \infty$  forces  $z^* = 0$ . Also by construction we have

$$1 \leq \frac{\overline{\Pi}(z_{n+1})}{\overline{\Pi}(z_n-)} = \frac{\overline{\Pi}(z_{n+1})}{\overline{\Pi}(z_n)} \left( \frac{\overline{\Pi}(z_n)}{\overline{\Pi}(z_n-)} \right) \leq 1 + \frac{1}{n}.$$

By (19) we have

$$\lim_{n \rightarrow \infty} \frac{\overline{\Pi}(z_n)}{\overline{\Pi}(z_n-)} = 1.$$

Therefore we get (21).  $\square$

According to Proposition 1.10.5 in [2] to establish that  $\overline{\Pi}$  is regularly varying at zero it suffices to produce  $\lambda_1$  and  $\lambda_2$  in  $(0, 1)$  such that for  $i = 1, 2$

$$\frac{\overline{\Pi}(\lambda_i z_n)}{\overline{\Pi}(z_n)} \rightarrow d_i \in (0, \infty), \text{ as } n \rightarrow \infty,$$

where  $(\log \lambda_1) / (\log \lambda_2)$  is finite and irrational. This can clearly be done using (20) and  $\mathbf{P}\{Y_k < 1\} > 0$ . Necessarily  $\overline{\Pi}$  has index of regular variation parameter  $-\alpha \in (-\infty, 0]$ . For  $\alpha \in (0, \infty)$  the limiting distribution function

has the form (4). In the case  $\alpha = 0$ ,  $\overline{\Pi}$  is slowly varying at 0 and we get that  $G_k(x) = 1$  for  $x \in (0, 1)$ , i.e.  $W_k = 0$  a.s.

Now consider the case when  $\mathbf{P}\{Y_k = 1\} = 1$ , i.e.  $G_k(x) = 0$  for any  $x \in (0, 1)$ . We once more use Theorem 1.7.1 in [2] with  $c = 0$  this time, and as an analog of (17) we obtain

$$\lim_{u \rightarrow \infty} \frac{\overline{\Lambda}(x\psi(u))}{u} = \infty.$$

This readily implies that

$$\lim_{z \downarrow 0, z \in \mathcal{A}} \frac{\overline{\Lambda}(xz)}{\overline{\Lambda}(z)} = \infty,$$

from which  $\overline{\Lambda} \in \text{RV}_0(-\infty)$  follows along the same lines as before.

**Sufficiency.** Assume that  $\overline{\Pi}$  is regularly varying at 0 with index  $-\alpha \in (-\infty, 0)$ . Then its asymptotic inverse function  $\psi$  is regularly varying at  $\infty$  with index  $-1/\alpha$ , therefore simply

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \rightarrow \left( \frac{S_k}{S_{k+1}} \right)^{1/\alpha} \quad \text{a.s., as } t \downarrow 0,$$

which has the distribution  $G_k$  in (4). Assume now that  $\overline{\Pi}$  is slowly varying at 0. Then  $\psi \in \text{RV}_\infty(-\infty)$ , therefore

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \rightarrow 0 \quad \text{a.s., as } t \downarrow 0.$$

Finally, if  $\overline{\Pi} \in \text{RV}_0(-\infty)$  then  $\psi$  is slowly varying at infinity, so

$$r_k(t) = \frac{\psi(S_{k+1}/t)}{\psi(S_k/t)} \rightarrow 1 \quad \text{a.s., as } t \downarrow 0,$$

and the theorem is completely proved.

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